ON MELLIN TRANSFORM

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Abstract. Zagier defines a generalized Mellin transform in [1]. The algebraic meaning is suggested via a comparison with Riemann-Hilbert problem, a la Connes-Kreimer.

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1. Generalized Mellin Transform

The generalized Mellin transform is defined in [1] using asymptotics of Dirichlet series. The Mellin transform is the multiplicative version of Fourier transform via exponential, while Dirichlet series is the analog of Fourier series.

The correspondence is via the exponential:

\[
\begin{align*}
(C, +) & \xrightarrow{e^t} (C^\times, \cdot) \\
& \downarrow t \\
(C, +) & \xrightarrow{e^t} (C^\times, \cdot) \\
j & \downarrow \gamma \\
(2) & = 1 \\
\end{align*}
\]

with Fourier transform “living” on the additive up-side (fiber), and Mellin transform on the multiplicative “down-side” (base) of the covering map.

The two sides are also related by the bicharacter / representation:

\[
\chi : (C, +) \times C^\times \to C^\times, \quad \chi(s, z) = z^s,
\]

\[
\chi : (C, +) \to \text{End}(C^\times, \cdot), \quad s \mapsto \chi_s.
\]

The asymptotics at zero and infinity can be formulated using the inversion \( J \) of the Riemann sphere \( \mathbb{P}^1 \mathbb{C} \), which corresponds to the symmetric space symmetry \( s \to -s \).

The multiplicative group \( C^\times \) has boundary \( \{0, \infty\} \).

The Poisson summation formula \( \sum_n f(n) = \sum_n \hat{f}(n) \) can be interpreted (and proved) using the “orbit integral”

\[
Z_+(f)(x) = \sum_{n \in \mathbb{Z}} f(x + n),
\]

Date: September 6, 2014.
for $f$ some Schwartz function on $(\mathbb{R}, +)$, in the context of (additive) Pontryagin duality
\[ \mathbb{Z} \to \mathbb{R} \to \mathbb{T}. \]
The multiplicative analog usually defined in connection with the exact formula a la Weil (a version of Riemann-Mangoldt exact formula using distributions) is
\[ Z. (f)(x) = \sum_{n \in \mathbb{N}} f(nx). \]
The “full” analog should be defined on the multiplicative group $\mathbb{Q}_+$:
\[ Z. (f)(x) = \sum_{r \in \mathbb{Q}_+} f(rx), \]
which requires that $f$ is rapidly decaying at infinity and zero.

Here we think of the rationals as part of the “projective line” of integers (the Riemann rational sphere):
\[ \mathbb{Q}_+ \subset \mathbb{P}^1 \mathbb{Z}. \]
In relation with the adelic picture, it is useful to view the rationals as functions with compact support on the set of primes $\text{Hom}_c (\mathbb{F}, \mathbb{Z})$ (or “divisors”: $\mathbb{Z}$-linear combinations of generators $X_p$ [2]).

The extension of Mellin transform via the two charts of the Riemann sphere has lots of similarities with the Birchoff decomposition used to define the finite part of Feynman path integrals.

Path integrals can be used to define Riemann surfaces, as an alternative tool to the classical analytic continuation (multivalued functions as path integrals vs. universal covers method to eliminate branching points).

More concretely, there should be a direct relation between this method and regularization methods for infinite products, for meromorphic functions:
\[ \text{Hadamard product } f(s) = e^{A_0 + A_1 s + \cdots} \prod (1 - z/\rho) e^{-z/\rho} \]
since zeroes (and poles a like, via logarithmic derivative) almost determine the meromorphic function (modulo Riemann-Roch Theorem).

From the physics point of view (conformal geometry as a 2D Electromagnetism; Cauchy Theorem as the combined Green’s Theorems), a meromorphic function is a free flow without divergence and curl, with sources at the boundary, e.g. at critical points (zeroes / poles).

2. Questions and Applications to Exact Formulas

In view of the above relationships, the generalized Mellin transform construction leads to some natural questions.

1) How does the bicharacters $z^s$ as “sources of rotations” and “divergence” (scaling) combine with “charges” (Dirichlet series) to integrate to such a flow (Mellin transform / Perron’s Formula)?

2) What is the role of asymptotics expansions at zero and infinity?

3) What is the role of Euler product, when restricting to the algebraic framework (p-adic numbers).
4) What are the implications for multiplicative duality and its important consequence:


It implies the exact (trace) formulas, which in the additive case is a consequence of Pontryagin duality Pontryagin, while in the multiplicative case, is unknown (primes vs. zeroes correspondence). Interpreted as a trace formula it equates the arithmetic/geometric side (e.g. sum of diagonal entries) to the spectral / harmonic side (sum of eigenvalues).

**Example 1**: (additive) Poisson Summation Formula (FT=Fourier transform, FS=Fourier series):

\[ FT(Z_+(f))(0) = 1 \cdot Z_+ (FT(f))(0), \]

assuming \( Z \) and \( FT \) commutes \(^4\).

**Example 2**: (multiplicative) Exact Formula \(^3\), p.5:

\[ MT(Z_\cdot(f))(0) = DS(1) \cdot MT(Z_\cdot(f)), \]

where MT denotes Mellin transform, DS=Dirichlet series (related to Discrete Mellin Transform: \( DS(1) = DMT(Id) \)).

Any comments from the reader are more then welcomed.

**REFERENCES**


[2] L. Ionescu, lecture notes on algebraic quantum groups, web; (see also Van Daele work).


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\(^4\)Is the trivial factor \( FS(1)(0) \), with \( FS(1) = \sum_{n \in \mathbb{N}} x^n = \frac{1}{1-x} \)?