Research on the Structure of Prime Numbers
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Abstract
The proposed research will investigate the **lattice structure of prime numbers**, recently discovered by the PI [18]. This discovery resulted from non-traditional investigation of the basic finite fields $F_p$ from a geometric perspective, and obtaining a partial order on their sizes, i.e. prime numbers, coming from their complexity, measured by their symmetries (generators). This idea of using “categorification” to study a mathematical structure is a common theme / tool in the research of the PI, as well as in modern mathematics (Khovanov cohomology, Topological Quantum Field Theory etc.). Using categorification in Number Theory is up-to-date a novelty, with deeper implication in Multiplicative Number Theory and other unsolved problems of Number Theory, like the Riemann Hypothesis, which are still approached in a traditional way using analytical methods [26].

The lattice structure gives additional “dimensions” to study the apparent chaotic occurrence of primes within the natural numbers. One important evidence of usefulness of this new structure would be proving in a direct manner the Prime Number Theorem, which is a central celebrated result in Number Theory, regarding the distribution of prime numbers on the discrete “line” of integers.

In this direction, the relation between the structure of primes and rooted trees will be studied. So far the PI did not publish his recent findings, waiting for a significant application of the new structure discovered, and leading to some “concrete” result. Such a result is expected as the main outcome of the proposed research, if funded.

1. Introduction
There are numerous indications that **Fundamental (Quantum) Physics**, at its foundations, is algebraic Number Theory [1]-[6], [15], [22], [25]. At the core of algebraic Number Theory and its Physics applications, are the prime numbers [1-4], [8-11], [18] (and references within). Moreover, understanding the **distribution of prime numbers** is a well-known difficult problem, without satisfactory “simple” proofs. A deeper understanding of prime numbers is essential for proving the celebrated Riemann Hypothesis, since the non-trivial zeroes of the Riemann zeta function encode the distribution of prime numbers [9-11], [20, 21].

2. Goals of the Proposed Research
Goal 1) Study the new lattice structure of the prime numbers, discovered recently by the PI [18], which comes from the structure of the basic finite fields $F_p$ (cycles with a prime number of nodes; rings of moduli $Z/pZ$).

Goal 2) Use the lattice structure to prove in a direct manner the Prime Number Theorem, which regards the distribution of prime numbers within the natural numbers (basic finite fields linearly ordered by size).

3. Significance of Research
Prime numbers are natural numbers which cannot be factored within the system of natural numbers. They are the “building blocks” for the integers. The PI discovered a “hidden” structure of prime numbers, coming from their “internal symmetries” once viewed as finite fields $F_p$ (number systems).

The above structure of the prime numbers, to be explained further on in this proposal, is new, being discovered by the PI [18]. The general view is that prime numbers appear in a chaotic manner, when compared with the linear order of the integers (1, 2, 3 etc.). There is a suspected relation with an unknown quantum dynamical system, which would explain the Riemann hypothesis. Understanding the partial order coming from this new structure of the prime numbers would allow further progress in the direction of the Riemann hypothesis. The latter is one of the most intriguing and famous conjectures in modern mathematics.

The study of the structure of the prime numbers has been the subject of a joint work with honor students, part of Mathematical-Physics research project at ISU Summer Research Academy 2011 [16].
The relation between number theory and physics, via a probability interpretation, was part of an independent study conducted by the author with the undergraduate student R. Bernales [15].

The proposed research has elementary components at the undergraduate and graduate student level. It also involves a computer exploration component using Mathematica as a computer algebra system.

The proposed research will also enlarge the applications to physics, especially to a discrete version of path integral formulation [7], via Mobius inversion principle and the incidence algebra [8].

The structure of prime numbers is not only of general interest, but can have a deep impact on cryptology, via the current encryption methods (e.g. RSA).

The proposed study has an important impact on to the topic of path integrals, which is a major theme in PI's research; it is also a natural continuation of the joint work with Prof. Fusun Akman's regarding the topic of partial ordered sets and incidence algebras, topics which has been the subject of several presentations in the Algebra Seminar. The PI noticed at that time important connections between number theoretical aspects and physics aspects, which now need to be made precise and reported to a wider audience, in the form of publications (see also [2]).

This research is also of potential interest to the discrete mathematics group in the department, generation many interesting problems formulated in terms of discrete mathematics.

4. The Plan to Achieve the Goals

4.1 The Partial Order on the Set of Prime Numbers

The point of view adopted is geometric. In a nutshell, a prime number p defines a primary finite field $\mathbb{Z}_p$ whose symmetries (automorphisms) define an orbit structure on $\mathbb{Z}_p=\mathbb{Z}/p\mathbb{Z}$. This reflects in the usual formula $p-1=q_1^{e_1}...q_r^{e_r}$ relating a prime p, i.e. the number of elements of the cycle $\mathbb{Z}_p$, with the factorization of p-1, i.e. the number of elements of the multiplicative group of its units, and representing the group of symmetries of the cycle $\mathbb{Z}_p$. This formula is reminiscent of the so called Proth numbers $N=2^k n + 1$, $2^k > n$ [12-14], and will be called the Proth decomposition.

The additional explanations which follow are taken from [18].

Natural numbers are “shadows” of finite abelian groups $\mathbb{Z}/n\mathbb{Z}$, denoted here $\mathbb{Z}_n$ (congruence classes modulo n form a number system widely used in cryptography for example), and the prime numbers correspond to finite fields $\mathbb{Z}_p$. There is considerable structure “lost” when passing from the algebraic structure to the numerical aspect via counting the number of elements (from “structure” to “size”).

For example, multiplication of integers cannot distinguish between multiplication of relatively prime numbers $p\cdot q$, which corresponds to direct sums $\mathbb{Z}_p\times\mathbb{Z}_q$, and powers of prime numbers $p^k=p\cdot...\cdot p$ which correspond to semi-direct sums of abelian groups.

If we look at $\mathbb{Z}_n$ as a space of a Klein geometry under the action of $Aut(\mathbb{Z}_n)$, then the primes reveal an interesting structure via their decomposition into disjoint orbits.

For example, $\mathbb{Z}_7$ has an orbit structure due to the factorization of its symmetry group with 7-1=6 elements, into $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

The PI therefore introduced a natural concept: the hierarchy rooted tree of a prime number p is constructed by iterating the Proth decomposition. If $p-1=q_1^{e_1}...q_r^{e_r}$ then the root has the trees associated to $q_i$ as descendants, with multiplicities $e_j$ (see the examples to the right).

In general, an arbitrary tree will not always represent a prime number. For example $2^3 7^2 + 1=393=3\cdot 131$ is not prime.

The hierarchy trees yielding prime numbers are called prime trees. As a special case, the prime trees representing Fermat primes, which are prime numbers of the form $2^k + 1$, where $k=2^n$, will be called Fermat trees. These are trees with one node $2^n$. The above defined trees generalize the proth primes [12], which are primes of the form $2^k\cdot n + 1$, with $2^k > n$.
additional examples were computed as part of a research project with students of the ISU Summer Research Academy [16]. For additional info on large primes of the form $2^k n + 1$ see [13, 14].

Now, prime numbers should not be considered in isolation: they are part of a partial ordered set, since the hierarchy trees will have terminals the prime 2. This new structure on the set of prime numbers is like a 2D-resolution of the usual linear 1D-order on the prime numbers derived from the order of natural numbers. The importance of the additional information revealed is similar to the difference between a TV-image flattened as a horizontal line, as opposed to the actual 2D-image.

4.2 Research Questions and Plan of Investigation
There are plenty of questions emerging: 1) which hierarchy trees yield primes? 2) how often such trees yield prime numbers? 3) Is it often enough to pose a threat to the RSA encryption scheme? etc. The first, is the major question regarding the primality of numbers of the form $N=2^k n+1$. These “primality tests” are criteria for testing if a natural number is prime or composite [13], p.1333.

A natural avenue of research is to try to generalize the so called Proth Theorem, which is such a primality test, to trees. There are other results which can be conjectured to hold for trees; for example the converse of Fermat's Theorem [19] will be investigated in terms of the direct relation between the factorization of $N$ and the properties of the trees corresponding to its prime factors.

The investigation of the properties of the Partial Ordered Set of primes will be carried at two levels:

A) Investigation of the results related with the “converse” of Fermat’s Theorem [18, 19];
B) Experimental investigation using the computer algebra system Mathematica, with possible help from students interested in number theory (ISU SRA [16], Master projects, independent studies etc.).

4.3 Relation with Past Research of the PI
The PI's research on the connections between Number Theory and Quantum Physics is a natural next step in the PI's research program. As mentioned in the introduction, the modern view of Quantum Physics emphasizes the discrete models, abandoning the “old” approach via Schrodinger Equation and Quantum Field Theory. Regarding the proposed research, several remarks have already been made by the PI [18], based on a thorough investigation of the current literature on the subject [20-25].

5. Conclusions and Further Developments
The proposed research targets some fundamental questions regarding prime numbers:

1. What is the role of the symmetries associated to a prime number (field) in conveying its property of being irreducible?
2. What is the structure of the partial order of the prime numbers? For example, the PI will prepare a description of an algorithm so that students will be able to “map” this new “territory”, using numerical investigations with computer algebra systems, as part of future independent studies and mathematics projects for the ISU Summer Research Academy.
3. How does this 2D-partial order impact the current statements about prime numbers which are usually investigated in the context of the total/linear natural order? For example, what is the distribution of primes as elements of the POSet, rather elements of the natural numbers?

The answers to the above questions have important connections with Riemann Hypothesis, Physics and cryptology. The concrete tasks involve experimental investigations using computers, an activity which may provide interesting projects for students. The topic is also of general interest to the faculty from the Discrete Mathematics group, within the ISU Mathematics Department. It potentially could lead to collaborations and presentations in the seminars.

The PI intends to write and submit research papers out of this investigation. The PI will also seek external funding by writing the upcoming summer an NSF proposal to be submitted Fall 2014.