Graph Complexes in Deformation Quantization
and
The Feynman Legacy
(Past, Present and Future)

by

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Talk based on the joint work with Domenico Fiorenza
math.QA/0505410 v.2

Bibliography:
• *Perturbative QFT and $L_\infty$—algebras*, hep-th/0307062,
preprint based on Kontsevich’s preprint q-alg/9709040:
• *Deformation quantization of Poisson manifolds, I*. 
Contents

The Feynman Legacy

• (Past) **Deformation Quantization:**
  Flato, Kontsevich, Cattaneo-Felder, etc.

• (Present) **Renormalization and Graph Homology:**
  Connes-Kreimer, Kontsevich, Fiorenza-Ionescu, etc.

• (Future) **Feynman Processes and Information Flow:**
  Fiorenza-Ionescu, B. Coecke, B. J. Hiley ...

  (Open list :-)
Why deformation quantization? (Moche Flato etc.)
- Start from the classical Poisson algebra of observables, and implement the Heisenberg commutation rules as a non-commutative deformations of the product:

\[ \star = \cdot + \hbar \{, , \} + \ldots \]

- It is a “conservative approach” initiated by M. Flato etc., disregarding Heisenberg’s message: model states and transitions (reiterated by Feynman).

Q: How to build a star product? A: Kontsevich’s formula:

\[ f \star g = \sum_{\Gamma} W_{\Gamma} B_{\gamma} \]

This is a series of differential operators \( B_{\Gamma} \) associated to graphs \( \Gamma \) according to a rule similar to a Feynman rule of QFT (perturbative approach):

\[ \mathcal{U} : \mathcal{G} \rightarrow CE(T_{\text{poly}}, D_{\text{poly}}) \]

\[ A = C^\infty(M), \text{ functions/observables on } M, \]
\[ \mathfrak{g} = \text{Der}(A) \text{ vector fields}, \]
\[ T_{\text{poly}} = \wedge^\cdot \mathfrak{g} \text{ polyvector fields}, \]
\[ D_{\text{poly}} = \text{Hoch}^\cdot(A; A), \text{ differential operators on } M. \]
• The mapping $\mathcal{U}$: associates to a graph $\Gamma$ colored by polyvector fields a differential operator:

$$< \mathcal{U}_\Gamma(\xi_1 \land \ldots \land \xi_n)|f_1 \otimes \ldots \otimes f_m > \in A.$$ 

• The bidifferential operators $B_\Gamma$: are obtained in the special case $m = 2$, when the graphs with two legs are colored using the Poisson bivector field $\pi$;

• Example - the 1st Bernoulli graph:

$$B_{b_1} = \mathcal{U}_{b_1}(\pi) = \pi.$$ 

• Kontsevich coefficients $W_\Gamma$: are obtained in a similar manner, using the same graphs, and a Feynman rule with a closed 2-form $\alpha$ (derived from the angle-form) on the Poincare half-plane (hyperbolic unit disk);

Amazingly, the formula works! (provides an L-infinity morphism and a star-product etc.) ... but why?
Why The Formula “works”?  

- On the “physics side”, this result (deformation quantization), is the outcome of a deeper theory: Feynman Path Integral quantization!  

The String Theory / sigma model interpretation was reviled by Cattaneo-Felder math.QA/9902090 (FPI quantization as a string theory/ sigma model on the Poincare half-plane).

- On the “mathematics side”, this result (formality) hides a deeper algebraic structure: dg-coalgebra (Hopf) of graphs, or DG-Lie algebra and morphism, Maurer-Cartan equation etc.

This was partially revealed by L.I. in hep-th/0307062, and explained in a “round form” together with D.F. in math.QA/0505410:

1) $U$ is a morphism of DGLAs, $W$ is a 1-cocycle of the DG-coalgebra of graphs (MC-solution of the dual DGLA), so $F = Wu$ is an L-infinity morphism.

2) $F = Wu$ is an $L_\infty$-morphism ($W$ is a DG-coalgebra cocycle/ DGLA MC-solution);

3) The Poisson structure $\alpha$ is a MC solution, which is mapped by the $L_\infty$-morphism $iF$ to a MC-solution, providing the deformation of the commutative product:

$$\star = \mu + F(e^\alpha).$$

It seems (to me) that the “underlying ideas” (algebraic structures involved) are more interesting then the “result” itself!

1It’s a “tautology”: quantization leads to quantization!
The Homological Algebra Interpretation

(Past)

• L.I.’s “IHES period” (working on renormalization and deformation quantization at the same time ²)
  - Kreimer’s coproduct and Kontsevich’s proof (sums of products ... must be a coproduct!)
  - “Reverse engineering” Kontsevich’s result ³: there’s a coproduct $\Delta$ and $W$ is a cocycle: $W(\Delta \Gamma) = 0$.
  - Step 1: Reformulating Kreimer’s coproduct, and applying it to Kontsevich formula;
  - Step 2: Kontsevich differential $d$ as a part of the coproduct.

• Conclusions (hep-th/0307062):
  1) Joining Kontsevich homology and Kreimer coproduct, yields a DG-coalgebra (Hopf) (dual picture: DGLA);
  2) Kontsevich coefficients $W$ form a corresponding coalgebra 1-cocycle (dual picture: MC-solution);

• Developing the theory: “turn On” the homological algebra machinery (cobar construction $D = d + \Delta$ etc.), yields graph cohomology (in particular, defines the Cohomology of Feynman Graphs: math.QA/0506142).

² How to catch two rabbits at a time: merge them into one :-)  
³ Concepts, concepts, concepts!
Solving “The Puzzle” (Together with Domenico Fiorenza)

- What is the map $\mathcal{U}$? It is a DGLA morphism mapping the Maurer-Cartan solutions in the DGLA of graphs to the Eilenberg-Chevalley complex.

- $\mathcal{U}$ maps the MC-solution $W$ to a MC-solution $U = \mathcal{U}(\sum_{\Gamma} W_{\Gamma})$.

Such an MC-solution is an L-infinity morphism and also a quasi-isomorphism (degree 0 part is) (Kontsevich’s proof of the Formality Conjecture).

Main results (math.QA/0505410): a conceptual interpretation of Kontsevich’s solution of the formality conjecture and deformation quantization of Poisson manifolds.

1) The Kontsevich graphs $\mathcal{G}$ have a DG-coalgebra structure, or equivalently a (dual) DGLA structure ($\S$5);

2) The Kontsevich graphical calculus $\mathcal{U}$ is a DGLA morphism ($\S$6) (calculus of derivations);

3) If $W$ is a cocycle of $\mathcal{G}$ ($\S$7), $\mathcal{U}(W)$ is an L-infinity morphism (the formality morphism) ($\S$4);

4) $\mathcal{F}(\exp(\pi))$ is a star product ($\S$4);

5) There is a solution involving only graphs without circuits ($\S$8): a semi-classical star product.
Main results explained

- The DG-coalgebra of Feynman graphs (§5):
  $\mathcal{G}^{\bullet, \bullet}$ is a bigraded DGLA, associated to a pre-Lie algebra of graphs, or dually a coalgebra, with coboundary differentials (internal differential is Kontsevich’s homology differential, and the external differential is Hochschild differential corresponding to Gerstenhaber composition by a preferred element)

- The DGLA morphism $\mathcal{U}$ (§6):
  the Kontsevich rule defining $\mathcal{U}$ (“a la Feynman”) is a graphical calculus of derivations (§2): vector fields $\partial_i$ act on functions $f$
  \[ X \mapsto \frac{\partial}{\partial x_i} f \]

- The formality morphism (between DGLAs):
  $\mathcal{F} = W\mathcal{U} : T_{\text{poly}} \to D_{\text{poly}}$
  is an L-infinity morphism, because $W$ is a cocycle in the DG-coalgebra of graphs (solution of the MC of the dual DGLA of graphs).
  Since its zero degree is known to be a quasi-isomorphism (Kostant-Hochschild-Rosenberg Th.), it proves the $D_{\text{poly}}$ DGLA is formal (quasi-isomorphic to its cohomology).
- The DG-coalgebra cocycle $W$:

$$0 = (D^*_{cob} W)(\Gamma) = W(d\Gamma + \Delta_b \Gamma)$$

$$= \sum_{e \text{ internal edge}} \pm W_{\Gamma/e} + \sum_{\gamma \subset \Gamma \text{ boundary}} \pm W_{\gamma} W_{\Gamma/\gamma},$$

or the **Maurer-Cartan solution** in the dual DGLA:

$$= \delta W + \frac{1}{2}[W, W](\Gamma).$$

It is defined using a similar “Feynman rule” with (winding number/angle form) $\alpha = d\theta$ defining the bivector field $dA = d\theta \wedge d\theta$.

- The **star product**: a MC-solution $\mathcal{F}$ in the CE-complex:

$$\text{Hom}(T^\bullet(T_{\text{poly}}, D_{\text{poly}}) \cong \text{Hom}(T^\bullet(\mathfrak{g}, D_{\text{poly}}),$$

for example $W U$, maps the exponential of a Poisson structure, i.e. a MC-solution in $T_{\text{poly}}$ (with trivial differential):

$$d\pi + [\pi, \pi] = [\pi, \pi] = 0 \quad (\text{Jacobi identity}),$$

to a MC-solution in $D_{\text{poly}}$, yielding a **perturbation** $\partial$, of the commutative product:

$$\star = \mu + \partial, \quad \partial = \mathcal{F}(\exp(\pi)).$$

**MC – solutions** $= T_\mu(\text{Moduli space of deformations}).$

- **Reduction to graphs without circuits**: if considering graphs without circuits, forests $\mathcal{F}$, the results hold essentially because the inclusion $\mathcal{F} \subset \mathcal{G}$ is a quasi-isomorphism of complexes. $^4$

$^4$A different cocycle $W$ may also solve the problem: math.QA/0507053.
Feynman Processes (a brief philosophical intermezzo)

- The method of Feynman Path Integrals, is a way of thinking: states and transitions, i.e. automata. It can be applied to classical mechanics as well as to QM/QFT (QM=(0+1)-QFT).

- Motto: Quantum Physics IS Quantum Computing (Point of view adopted by Feynman himself).

- Quantum interactions are Quantum Communications: enables a unified Mathematical-Physics and Computer Science description, without the asymmetry $system \rightarrow observer$, towards the incorporation of entropy and information as part of the foundations of physics (An extended Equivalence Principle between energy and information; e.g. the “unit of a black-hole surface” is a bit of space-time - Lee Smolin; more general, quantum BH are “generic”: spin/qubit networks etc.).
Feynman Processes
(The Mathematical-Physics)

- **Representations** of a **causal structure**:
  - **Geometric category** (playing the role of “space-time”; NO embeddings yet!) e.g. Feynman-Kontsevich graphs, Segal PROP, cobordism categories, lattices etc.
  - **Functorial representation** (algebra over the PROP): QFT, CFT, ST, TQFT etc.

- **Physics interpretation and motivation (QG):**
  - In a **causal structure** there may be no locally defined space-time structure! (There may be loops etc.).
  - The causal structure is **NOT “fixed”** (like a manifold), but it is a **resolution** (role of multi-scale analysis, Haar wavelets MRA etc.); the **processes** are inter-related as a **complex**;
  - It is **NOT “the perturbative approach”**; but it may be the **outcome** of one:

  \[ Z = \int \mathcal{D}\phi e^{S(\phi)} \quad \overset{\text{Wick's Th.}}{\longrightarrow} \sum_{\Gamma} \mathcal{F}(\Gamma)/|\text{Aut}(\Gamma)| h^{\text{deg}(\Gamma)} \]

  **FPI and matrix-models** are **“recipes”** for representations of causal structures. \(^5\)

\(^5\)Compare: DE and \( e^x \) - Taylor expansion ⇒ generating function of \( |\text{Aut}[n]| \) (“perturbation”: smooth → analytic/combinatorial/algebraic).
Abstract groups encoding the algebraic structure of groups of transformations (representations).

- **Operads**: encoding *types of algebraic structures* (like abstract groups encoded the “common practice”: groups of transformations):
  - Formal definition (D.F./Intro. to operads): a $C$ operad over $N$: $O(n) \in Ob(C)$ (“operations”) and rules for composing these operations (the In/Out-motherboard picture);
  - **Ideals** and **quotients** (see Examples)
  - Presentations by **generators and relations** (see Examples)

- **Examples of operads**:
  - The operad $\text{Assoc}$ (associative algebras; binary trees and relations);
  - The operad $\text{Lie}$ (Jacobi id.)
  - The operad $\text{Comm}$ (commutative algebras)

- **Representations of operads**: algebras over an operad $\rho: O \to \text{Vect}_k$.
  The endomorphism operad $\text{End}(V)(n) = \text{Hom}(V^\otimes n, V)$ etc.
“Doubling operads” ⇒ PROPs
- Multi-outputs and compositions: \( \mathcal{P}(m, n) \) (I/O)
- Operads become “half-PROPs”: \( \mathcal{O} \subset \mathcal{P} \), and the standard PROP generated by an operad \( \mathcal{P}(\mathcal{O}) = \mathcal{O}^*\mathcal{O} \)
- Examples:
  - The endomorphims PROP \( \mathcal{P}(n, m) = Hom(V^n, V^m) \).
  - **Feynman PROP** (Gerstenhaber composition),
  - **Cobordisms** (Topological gluing),
  - **Segal PROP** (Sewing Riemann surfaces) etc.

Algebras over PROPs:
- Representations \( \rho : \mathcal{P} \to Vect_k \) etc.
- Examples (see A. A. Voronov - hep-th/9401023):
  - **QFT** (Feynman rules),
  - **TQFTs** (e.g. Frobenius algebras)
  - **CFT** (sewing Riemann surfaces) etc.
  - **String Backgrounds** (chains of RS and homotopy Lie algebras): \( C_*\mathcal{P}(m, n) \to End(H, Q)(m, n), \ Q^2 = 0 \).

More structure: DG-coalgebra PROPs / homological algebra of PROPs (homology differential, insertion/elimination of subgraphs as extensions, cohomology of Feynman graphs etc.)

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6 also an algebra over the PROP of trees
“Perturbative” or not?
(This is the question ...)

- What is “new” in this “perturbative approaches” to “space-time”?  
  - Enables the resolution / multi-scale analysis (MRA);
  - Involves in a fundamental way “symmetry fluctuations” of the causal structure (and entropy etc.).
  - Prompts for an $Q$-information flow interpretation of the quantum dynamics (FPI at two levels: partition function AND within a possible Feynman graph/RS etc.)

- So, what is a Feynman process? Suggested by the DG-coalgebra structure of Feynman graphs

\[
\begin{align*}
\gamma \subset \Gamma & \rightarrow \Gamma/\gamma \\
\gamma \subset [k] & \overset{\gamma}{\rightarrow} [l] \\
\gamma \subset [n] & \overset{\Gamma}{\rightarrow} [m] \\
\gamma \subset [n-k] & \overset{\Gamma/\gamma}{\rightarrow} [m-l],
\end{align*}
\]

a Feynman Category (FC) is a **DG-coalgebra PROP** of finite type (may be with some additional structure ...).

The coalgebra structure encodes the factorization of morphisms/processes in the space-like/resolution depth direction.

A Feynman process is an algebra over a FC (involving an $SU(2) \oplus SU(2)$ action ...)

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7Better: it IS a model for a discrete (quantum) “space-time”, NOT just an approximation of a continuum one; the continuum limit is the approximation!

8Related to the Vir/conformal symmetries etc.
Homology & Cohomology of Feynman Categories
(How to build Feynman Processes)

- Once a Feynman Category $\mathcal{F}$ is given, one can study the **generalized (co)homology of a manifold**: embedding points, graphs, surfaces, cobordisms etc. (homology/homotopy of the configuration spaces):

  \[ \text{“Hom(geometric category, ambient manifold)"} \]

Example/model: the category of finite ordered sets $\Delta$.

- **Develop an abstract theory of FC** (from continuum/manifold theory to discrete / graph theory):

  1) Discretize the manifold;

  2) **Pullback “The Theory”** to the “geometric category” (e.g. history of abstract groups: groups of transformations to abstract groups);

  3) Study the representation theory (Cohomology with coefficients in a modular category).

- **Exercise**: start with Feynman-Kontsevich graphs as a warm up, before attacking String Theory!

- **QFT and graph cohomology, CFT, TQFT etc.: cohomological physics (Stasheff).**

- **Work in progress with Domenico Fiorenza**: Configuration spaces, cohomology of Feynman diagrams and Connes-Kreimer renormalization.
From *Continuum* to Discrete

- **The “pull-back philosophy” is a growing trend in high-energy physics:**
  - Loop QG starts from a GR manifold picture and ends up with a discrete space-time (spin-networks and spin-foams etc.);
  - String Theory as a “background-free” theory (future);
  - It saves time to “adopt” the “Feynman Picture” from the beginning. Feynman processes are “just” enriched and complexified Markov processes ... 

- **What is “Space-Time”?**
  - “It does not matter; all we need is a resolution!” (Paraphrasing Manin-Gelfand, *Homological algebra*)
  - The K’s sigma-model quasi-isomorphism IS a resolution of the “ambient space-time” 9;
  - *Feynman Process = Geometric Category* (“resolution of space-time”), σ-model $\text{Hom}(\cdot, M)$ (*Configuration Functor*) and **FPI-quantization** → its derived functors ...?
  - The importance of Kontsevich’s approach to deformation quantization based on graphs: **PROP with extentions**, like a *homotopy structure* (I don’t see the “extension capability” for Riemann surfaces; where is the differential?)

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9 Conjecture; to be made precise later ...
The “Missing Physics”...

- Equivalence between Energy and Information
  - Shannon entropy:
    \[ S = k \ln W \quad \iff \quad S = k |Aut(\Gamma)|. \]
  i.e. entropy as a measure of symmetry!
  - Feynman Processes as Quantum Information Dynamics (QID).
  - Equivalence between mass-energy and entropy (information), at the fundamental level: include entropy in the action, via the symmetry group:
    \[
    Z = \int_{\gamma \in \text{Hom}(\text{In,Out})} e^{iS(\gamma) / |Aut(\gamma)|}, \]
    \[
    e^{iS(\gamma) / |Aut(\gamma)|} = e^{-\ln |Aut(\gamma)| + iS(\gamma)}
    \]
    \[
    Z = \int_{\text{Hom}(\text{In,Out})} e^{S + iS} d\mu.
    \]

- Other connections (speculations):
  - Green functions and \( S + iS = -\ln |Aut(\gamma)| + iS(\gamma) \) etc.
  - Unifying Euclidean field theory and statistical formalism; generalizing Wick rotation to completely unify space and time;
    - Breaking/changing the symmetry (change in entropy) is at the same level with energy flow: are mass and gravity an entropic effect ... (rest mass/energy = entropy).
Additional Evidence:

- Laws of radiation of black holes (entropy is proportional with the surface area etc.)
- Other contributions hinting in the “same direction”:
  - Lee Smolin - “pixel of space-time” (perhaps better: qubit);
  - B. J. Hiley and quantum potential (I add: information potential)
  - Bob Coecke and quantum information-flow at the level of QM, meaning the quantum computation “order” (flow); (Must be generalized to QFT)
- Presenter’s personal impression: there is “new physics” at the horizon \(^{10}\), for the already existing mathematics (It wouldn’t be for the 1st time!)

Bibliography:


\(^{10}\)”Low Entropy Physics”: LEP versus HEP-th.”
Conclusions

- HEP-th is a study of representations of PROPs: Feynman, Segal, cobordism categories etc.

“New Mathematics”?

- A Feynman Category ("geometric category" representing the causal structure), e.g. Feynman graphs, is a resolution of “space-time” (Perturbative versus “Resolution” point of view: different grading)

- (Project) QFT, String Theory etc. and “derived functors” of a Configuration Functor (σ models)

- Importance of Kontsevich’s construction and quasi-iso: is the formality morphism an augmentation of a resolution of the sigma model by a causal structure?

\[
\text{Feynman Category} \xrightarrow{\text{quasi-iso}} \text{Formal Manifold}
\]

\[
DGCA/DGLA : (G, d, \Delta) \xrightarrow{\varepsilon} (\text{End}(T(A)), Q).
\]

“New Physics”?

- Entropy as a measure of symmetry enters the dynamics of the space-time (information dynamics in an extended sense);

- Feynman Processes as Quantum Information Dynamics (QID)

\[
\ldots (\text{The End } ^{11}) \ldots
\]

\[11 \cong \text{ new beginning.}\]