Ideals and Homology in Additive Categories

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I will present some results on homology in additive categories. This approach allows a generalization of the machinary of derived functors from exact categories to additve categories, and through linearization, to arbitrary categories and functors.

The details can be found in the (e-)preprints (preliminary versions) :

"Ideals and Homology in Additive Categories", math.CT/9906039 .

"On categorification", math.CT/9906038 .

2. **Outline**

- Ideals - the **intrinsic** approach.

Ideals are used as an intrinsic approach to extend some results from homological algebra which usually require exact categories, to the context of additive categories.

- Categorification - the bridge to ring theory.

A certain correspondence with ring theory is used. It is part of a more general correspondence principle, refered to as "categorification".

- Homology in additive categories.

To define homology one needs the fundamental universal constructions.

  - Universal constructions: Ker, Coker, etc. as ideals.

The familiar concepts of kernels, cokernels etc. may be replaced by considering the corresponding ideals, which are subfunctors of Hom.

  - "Homology groups" as functors (modules).

In this way to a complex in an additive category one can associate the quotients of these subfunctors, obtaining new functors (homology modules).

- Examples: Derived Categories.

Important examples are derived categories of abelian categories, which are not necessarily abelian. One would still want to have Ker, Coker, etc. in a generalized sense, and to study these categories using "abelian techniques", e.g. the machinary of derived functors.

- Representation theorem: reduction to ordinary homology in the abelian case.

When the category is abelian the new homology functors are representable by the usual homology groups.

- Derived functors of a nonadditive functor.

This approach allows one to generalize the construction of derived functors to arbitrary functors. Some preliminary results are obtained, but which are not included in the above cited preprint.

- Conclusions.

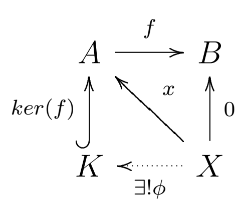
The relationship between nonabelian cohomology and this nonabelian, but still additive, homology algebra will be investigated.

3. **Ideals**

- Ideals - the **intrinsic** approach.

Universal constructions in category theory represent classes of morphisms in terms of a (universal) generator (a limit). The use of "coordinates" in geometry, or the use of generators and relations in algebra is opposed to the intrinsic point of view which emphasizes the "global object".

- An example - Kernels.



$ker(f)$ - universal morphism, unique up to "units".

Well-known diagram; univ. property; may or may not exist. As a limit (equalizer): infimum for a certain diagram.

A possible interpretation comes from a comparison with a classical situation.

- Categorification - bridge to ring theory.

It is well known that a group may be viewed as a one object groupoid (Mac Lane 1946), and a ring can be thought of as a one object additive category. The precise statement is that there is a functorial correspondence that we call the "tautological categorification", between the above mentioned concepts.

Group $<->$ One object groupoid (Eilenberg-Mac Lane, 1945).

Ring $<->$ One object additive category ("Rings with several objects", B. Mitchell, 1972).

- Kernels as annihilator ideals.

Through this correspondence, the morphisms whose composition with f is zero is in the one object category case (ring) a right ideal. The existence of kernels in the categorical sense means that this right ideal is principal and the universality property means it is free as a right module, i.e. that the one generator forms a base, and therefor it is unique modulo a "unit" (isomorphism):

$Ker(f) := \{ker(f) \circ \phi | \forall \phi\} = Ann_R(f) := \{x | fx = 0\}$

$ker(f)$ exists (categorical level) iff $Ann_R(f)$ is a free & **principal** right ideal (ring theory level).

What benefit we have from this reinterpretation? To define homology of complexes in a category, one normally assumes that there exists a null family of morphisms and kernels and cokernels exist. Let's consider the intrinsic approach to homology in the context of additive categories, using ideals.

4. **Ideals & Homology**

- Ideal in an additive category.

Ideals were introduced by G.M. Kelly in 1964. An ideal is a subfunctor of $Hom_{\mathcal{A}}$. It is essentially a class of morphisms closed under composition.

- Generalization of homology to additive categories corresponding to the "non-principal case".
  - Universal constructions - $Kerr(I)$, $Coker(I)$ as right / left ideals (I left/right ideal).

When left/right annihilators are not necessarily of "dimension 1", the familiar concepts of kernels, cokernels, image and coimage may be replaced by the corresponding ideals, which are subfunctors of Hom.

The corresponding ideals will be denoted with capital letters. Kernels and cokernels as ideals are naturally defined also for arbitrary left or right ideals, not only for morphisms (morphisms are viewed as principal ideals).

∘ $Im(f) = Ker(Coker(f))$, $Coim(f) = Coker(Ker(f))$ always exist.

As a first consequence, two of the basic axioms of an exact category are satisfied. Another imediate consequence is the *intrinsic* definition of homology, i.e. without a choice of a representative for the homology group.

Consider the complex:

$$...C_{n-1} \xrightarrow{d_{n-1}} C_n \xrightarrow{d_n} C_{n+1}...$$

Then $Im(d_n) \subset Ker(d_{n+1})$ (inclusion of right ideals). and "homology groups" are defined as functors:

$$0 \longrightarrow Im(d_{n-1}) \hookrightarrow Ker(d_n) \twoheadrightarrow H_n^R(C_\bullet) \longrightarrow 0$$

In this way to a complex in an additive category one can associate the quotients of these two subfunctors, obtaining new functors (homology modules), and replacing the usual noncanonical homology groups.

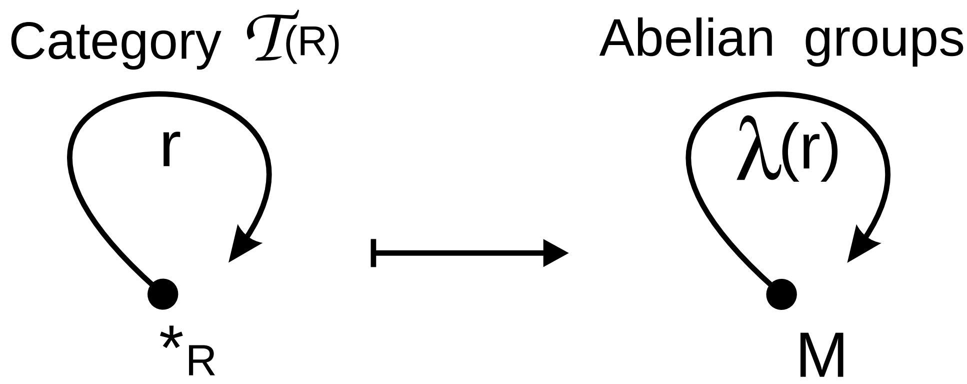
5. **Modules as Functors**

• Categorification of modules.

A set-theory R-module is a morphism from a ring to the endomorphism ring of an abelian group: $R \xrightarrow{\lambda} End(M)$ (morphism). Through the tautological categorification it is mapped to a functor:

$*_R \overset{\Lambda}{\mapsto} M$

$R = End(*_R) \overset{\Lambda}{\mapsto} End(M)$.



By considering the generic case one obtains the notion of a category theory module. It is an additive functor $\Lambda : \mathcal{A} \to \mathcal{A}b$ from an additive category $\mathcal{A}$ to the category of abelian groups.

6. **Homology Modules**

Let $\mathcal{A}$ be an additive category and $Comp(\mathcal{A})$ the category of complexes of objects in $\mathcal{A}$ with chain maps as morphisms.

Definition 6.1. The *homology modules* of a complex $C_\bullet$ in an additive category are:

$$H_n^R(C_\bullet) = Ker(d_n)/Im(d_{n-1}) \quad \text{(right module)}$$

$$H_n^L(C_\bullet) = Coker(d_{n-1})/Coim(d_n) \quad \text{(left module)}$$

i.e. the functors $H_n^R : Comp(\mathcal{A}) \to Hom(\mathcal{A}, \mathcal{A}b)$ and $H_n^L : Comp(\mathcal{A}) \to Hom(\mathcal{A}^{op}, \mathcal{A}b)$ with values:

$$H_n^R(C_\bullet)(X) = \{\phi : X \to C_n | d_n \circ \phi\}/ \\ \{\phi : X \to C_n | Coker(d_{n-1}) \circ \phi = 0\}$$

equivalence classes of morphisms (right annihilators of d_n) modulo a certain subclass of morphisms (right annihilators of left annihilators of d_{n-1}).

7. Relationship with usual homology

- The representation theorem.

When the category is abelian the new homology functors are representable by the usual homology groups.

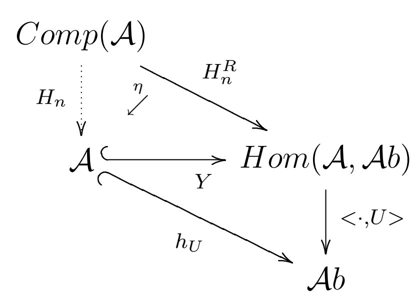
Theorem 7.1. *Let $\mathcal{A}$ be an abelian category and $C_\bullet$ a complex in $\mathcal{A}$. Then the homology modules are represented on projective objects by the homology groups:*

$$H_n^R(C_\bullet)(P) \underset{iso}{\overset{\eta_P}{\to}} Hom(P, H_n(C_\bullet)), \quad P \text{ projective}$$

In particular when the category has a projective generator U, then the category is canonically embedded in the category of abelian groups, and the two definitions of homology functors canonically correspond.

- Embeddings: Y (Yoneda embeding), h_U (U projective generator).

In the following diagram Y denotes the Yoneda embeding and h_U is the canonical representable functor which embeds the additive category $\mathcal{A}$ into the abelian category of abelian groups. Recall that a projective generator can be characterized by the additive functor h_U being exact and faithfull.



- Example: $\mathcal{A} = R - mod$ (R commutative ring).

Corollary 7.1. *If $U = R$ is the canonical projective generator of $R-mod$ then there are canonical isomorphisms:*

$$H_n^R(\cdot)(U) \cong h_U \circ H_n \cong H_n$$

This is essentially due to the identification of the elements of an R-module M with the morphisms $Hom(R, M)$.

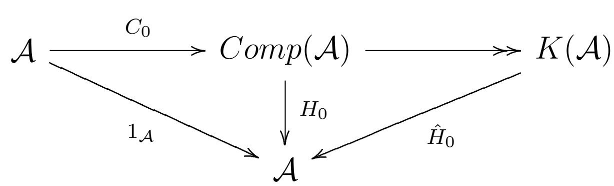
8. **Applications of Ker & Coker**

- Derived categories of an abelian category need not be abelian.

Important examples are derived categories of abelian categories, which are not necessarily abelian. One would still want to have the notions of kernel and cokernel, etc. in a generalized sense, and to study these categories using "abelian techniques", e.g. the machinary of derived functors.

Recall that, given an abelian category $\mathcal{A}$, one can consider the category of complexes and chain maps $Comp(\mathcal{A})$ which is also abelian.

Define the category $K(\mathcal{A}) = Comp(\mathcal{A})/\mathcal{I}$ with the same objects as $Comp(\mathcal{A})$ (complexes) and with morphisms **homotopy classes** of chain maps, i.e. classes of morphisms modulo the ideal $\mathcal{I} \subset Hom_{Comp(\mathcal{A})}$ of null chain homotopic chain maps.



In the above diagram C_0 embeds the objects and morphisms as complexes and chain maps concentrated in degree 0.

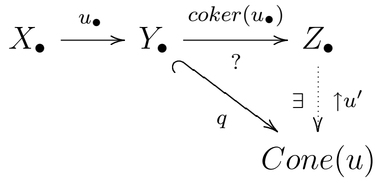
We will not need the actual derived category $\mathcal{D}(\mathcal{A}) = K(\mathcal{A})[\Sigma^{-1}]$, which is obtained as a localization with respect to quasi-isomorphisms.

- In general $K(\mathcal{A})$ is *not an exact category* (nor is $\mathcal{D}(\mathcal{A})$).

In the general case, the category $K(\mathcal{A})$, obtained by considering homotopy classes of morphisms, is no longer abelian. It may fail even to be exact, as shown by the next example.

9. **Example**

Explicitly, if $\mathcal{A}$ is a category with nontrivial extensions (e.g. abelian groups), and if $u \in Hom_{\mathcal{A}}(X, Y)$ is a morphism with its image $Im(u)$ not a direct summand in Y, then the corresponding morphisms in $Comp(\mathcal{A})$ (concentrated in degree 0) has no "classical" cokernel:



The key is that $q \circ u$ is chain homotopic to zero, and if u would have a cokernel, then the s.e.s. corresponding to its image (in $\mathcal{A}$) must be split ("Algebraic D-modules", A. Borel et. al., 1987, p.45).

Note that the *ideal* $Coker(u)$ always exists.

There may be applications to algebraic geometry, since $\mathcal{D}^b(Coh(P^n))$ corresponds to equivalence classes of algebraic vector bundles over P^n ("Algebraic bundles over P^n and problems of linear algebra", I.N. Bernstein, S.I. Gelfand, 1978).

Is this an alternative to *distinguished triangles*?

10. **Applications: derived functors**

- Further developments: derived functors.

Since the key lemmas "lift" from generators to ideals, homology theories can be defined for additve categories.

  - ○ Connected sequence of functors, etc..

There is an analog of the connecting transformation, which in the abelian case is represented by the usual connecting morphism.

Lemma 10.1. *Restricting to projective objects and morphisms, there are natural transformations* δ^R_n *such that* (H^R_n, δ^R_n) *is a connected sequence of functors.*

Similarly, the usual lemma leading to the machinery of derived functors holds.

  - ○ Chain homotopic morphisms.

Lemma 10.2. *Chain homotopic morphisms induce canonical isomorphims in homology:*

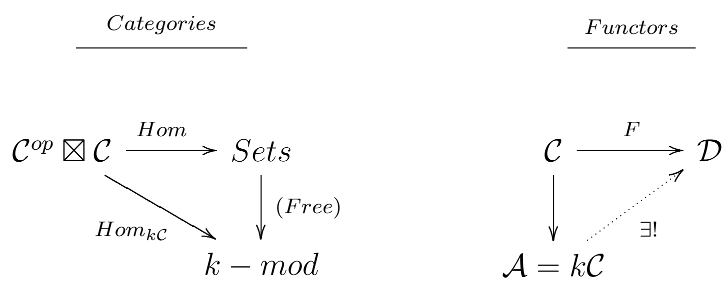
$$f_\bullet \sim g_\bullet \Rightarrow H_n^R(f_\bullet) = H_n^R(g_\bullet).$$

- Satellites of a nonadditive functor.

This approach allows one to generalize the construction of derived functors to the case of non-abelian functors. Some preliminary results are obtained, which are not included in the preprint math.CT 9906039.

More precisely, for any category one may consider the category with the same objects, and with $Hom(X,Y)$ the free module generated by the corresponding morphisms of the original category. Enriching the category in this way may be viewed as the categorification of the group ring construction.

Linearization of:



In this way one may apply the machinery of derived functors in the general case of an arbitrary category.

11. **Conclusions**

- Homology: from the "principal & free" to the general case.
  - ∘ $ker, coker$, etc.: generators of left/right principal ideals (subfunctors).
  - ∘ $H_n^R(C_\bullet)$ homology modules - functors.
  - ∘ If the category is abelian, the homology modules H_n^R and H_n^L are represented by the corresponding homology groups.

- Further developments and applications:
  - ∘ Derived functors (nonabelian case).
  - ∘ Nonabelian cohomology.

The relationship between nonabelian cohomology and this nonabelian, but still additive, homologycal algebra will be investigated.

- Derived categories in algebraic geometry.

Are there applications to derived categories in algebraic geometry? In what extent is it an alternative approach to distinguished triangles?