IN SEARCH OF A DUALITY

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Abstract. Exact formulas are consequences of duality: algebraic, at group level or group algebra, and analytic, via Fourier transform on $L^2$-spaces.

We compare the additive case with the multiplicative case, in search of the duality behind the Riemann-Mangoldt exact formula.

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1. INTRODUCTION

To understand Riemann-Mangoldt exact formula (or Weil’s form), referred to as the multiplicative case, and the relation with Selberg’s exact formula (Riemannian case), a comparison with Poisson Summation Formula as a trace formula, is in order.

2. THE CIRCLE: PONTRYAGIN DUALITY AND FOURIER TRANSFORM

The additive, real, 1-dimensional case is the case of the circle $G = T$ and its universal cover (equivariant picture):

$$\mathbb{Z} \rightarrow \mathbb{R}^\times T.$$ 

The short exact sequence (bundle) compares with the (Pontryagin) group duality

$$\langle, \rangle : \hat{G} \times G \rightarrow \mathbb{C}, \quad \ldots$$

The s.e.s. implies $\hat{T} \cong \mathbb{Z}$ (via a section $s : T \rightarrow R$, and its 2-cocycle? (rusty :) ... later)).

Passing from groups to rings is better: the duality induces an algebraic quantum groups duality between the group algebra and convolution algebra [1, 2]. It is the modern way to avoid the complications of tending (comultiplication etc.), and uses multipliers (essential in the multiplicative case: $L_n(x) = nx$ etc.)

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The classical analytical duality passes from group level to $L^1/L^2$-spaces and Fourier transform, although the above “group algebra to convolution algebra” is still present:

$$\text{FT}(f * g) = \text{FT}(f) \cdot \text{FT}(g).$$

There is a “famous” consequence: Poisson Summation Formula (reduced form):

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \text{FT}(f)(n),$$

where the dummy variable $f$ is a Schwartz function, suggesting an interpretation in terms of distributions, which extend the function-points duality:

$$L^2(G) \times G \to \mathbb{C}, \quad (f, g) \mapsto f(g).$$

The classical analytical way to deal with duality via $\text{Hom}(\cdot, \mathbb{C})$ contrasts the modern use of tensor products (monoidal structure), which is resolved via multipliers (see Van Daele), whenever $\otimes$ is an internal functor.

It is tempting to eliminate the “middle man”, the dummy variable $f$, “forcing” the analytic PSF into an algebraic form, by linearizing the function pairing or interpreting as distributions:

$$\langle f, \sum_{n} \delta_n \rangle = \langle \text{FT}(f), \sum_{n} \delta_n \rangle$$

where $\delta_n$ is the Dirac distribution

$$\langle f, \sum_{n} \delta_n \rangle = \langle f, \sum \text{FT}^*(\delta_n) \rangle$$

where $\sum g = \sum g'$. It is worth comparing with what happens in the discrete case [4] (later).

3. The Exact Formula as a Trace Formula

There are various accounts of the PSF as a trace formula [9, 5, 8, 7, 6] etc. The arithmetic side is a trace of an operator, and the geometric side is the sum of eigenvalues.

In the Selberg’s formula case (Riemannian geometry), the minimal geodesics ("homological side") are in "duality" (like Hodge duality?) with the eigenvalues of the Laplacian, as one would expect from a Hodge duality, where canonical harmonic forms represent cohomological classes, which under Poincare duality, correspond to homological cycles.

The better understood case is that of elliptic curves and Jacobi/Abel mapping, with the “clear” duality between homological cycles (choice of base) and dual 1-forms (Jacobian matrix etc.).

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$^1$The general framework: categories with duality, e.g. [3] and Yoneda representation.
4. The Complex "Circle": Elliptic Curves

From $Z \to R \to T$, we complexify and have:

$$Z[i] \to (C, +) \xrightarrow{\exp(z)} (C^\times, \cdot),$$

or better considering a general lattice $\mathcal{L}$:

$$\mathcal{L} \to C \to (C)^\times$$

and the associated elliptic curve $\mathcal{E} = C/\mathcal{L}$ (torus as a "fat circle"; elliptic functions as 2D-trigonometry etc.).

The Abel-Jacobi map and Weierstrass coordinates $\mathcal{P}, \mathcal{P}'$ correspond via the exponential when taking the "cokernel":

\[
\begin{array}{c}
\mathcal{L} \\
\downarrow \\
(C, +) \\
\downarrow \text{mod } \mathcal{L} \\
\mathcal{C}/\mathcal{L} \xrightarrow{\text{Weierstrass}} \mathcal{E}
\end{array}
\]

\[
\begin{array}{c}
q^Z \\
\downarrow \\
\exp(C^\times, \cdot) \\
\downarrow \text{mod } q^Z \\
\mathcal{C}/\mathcal{L} \xrightarrow{\text{Abel/Jacobi}} \mathcal{E}
\end{array}
\]

where $q = \exp(it)$, with $\mathcal{L} = \langle 1, \tau \rangle$.

That's the group / s.e.s. duality level.

The group algebra / convolution algebra is more complicated (one day ... :)

The analytical ($L^2$-spaces) side turns into complex analysis here ... It would be instructive to see the relation between the distributional framework (Schwartz functions etc.) and Cauchy-homotopical duality:

$$f(z_0) = \int_C \frac{f(z)}{z - z_0} dz \iff <\delta_{z_0}, f>_{\text{distributions}} = <\frac{1}{z - z_0}, f>_{C},$$

as if a cycle $C$ defines a linear distribution (physics picture: point, line, surface distributions etc.).

The “signature” of the lattice, and hence of the duality (s.e.s.), is the Weierstrass $p$-function

$$\mathcal{P}(z) = \sum_{u \in \mathcal{L}^\times} \frac{1}{(z - u)^2},$$

which is a “Coulomb (harmonic) force”, related to a potential $1/z$ (Hodge / harmonic functions?).

It is obtained from the Weierstrass zeta function:

$$\mathcal{P}(z) = -\zeta'(z), \quad \zeta = \frac{d}{dz} \sigma(z) = \sum_{w \in \Lambda} \left( \frac{1}{z - w} + L(z; w) \right),$$

where $L(z; 0) = 0$ and $L(z; w) = w^{-1} + z \cdot w^{-2}, w \neq 0$ is a “renormalization” term, which derives from the Weierstrass sigma-function (Hadamard product form):

$$\sigma(z) = z \prod_{w \in \mathcal{A}^\times} (1 - \frac{z}{w})^e z / w + (z/w)^2 / 2!,$$
5. Multiplicative case: $\infty$-dim $p$-adic torus

Now the discrete group $(\mathbb{Q}_+, \cdot)$ can be viewed as an algebraic quantum group [16].

The corresponding duality is the adelic duality:

$$\mathbb{Q} \to A = R \oplus A^{alg} \to (\mathbb{Q}, +),$$

where $A^{alg} = \bigoplus_{p \in P} \mathbb{Q}_p$ (the fibered product over $\mathbb{Q}$) is the “algebraic part”.

The relation to zeta function is via its logarithmic derivative $^2$:

$$< E > = -\frac{d}{ds}(\ln \zeta) = \sum \lambda(n)n^{-s}$$

which looks like a Weierstrass zeta-function eq. 1 for some unknown “lattice of zeta zeroes” when expressed as [13], p.52:

$$< E > (s) - < E > (0) = < E > \bigg|_0^s = \left[ \frac{1}{x-1} - \sum_{\rho \in \text{Zeroes}} \frac{1}{x-\rho} \right]_0^s,$$

where the Zeroes consists in trivial and non-trivial zeroes. Since logarithmic differentiation eliminates multiplicity and “treats” zeroes and poles alike, the formula reduces to:

$$< E > \bigg|_0^s = \sum_{c \in \text{C.P.}} 1 - c^{-s},$$

i.e. it is the superposition of the “tangential maps” at critical points (?). It is also reminiscent of a potential function (some path integral from an exact form; ?).

**Remark 5.1.** An analytic function is a divergent and curl free flow, with its “topological sources” at the boundary, e.g. at the poles. Then the above formula corresponds to an index formula: $< E > / E_0$ is the sum of charges, where $E_0$ is the energy of the ground state, and $F = qU$ is the force / charge definition, with $U$ the potential energy via a path integral.

In this way, the Newtonian continuum model is related to the discrete quantum model, via “topological charges” (see also [14, 15]).

The relation with the lattice interpretation (from elliptic curves to adeles duality and algebraic quantum groups), is now apparent.

**Remark 5.2.** It is sometimes more natural to look at the “fermionic” zeta function, instead of the “bosonic” zeta function $\zeta^+ = \zeta(s)$:

$$\zeta^-(s) = DS(\mu), \quad \zeta^- \cdot \zeta(s) = 1^3$$

The multiplicative (quantum) duality should relate this “lattice” with the primes $P$, which is the basis in the Lie module of primes:

$$g_P = ZP^E \times P^Q^*.$$  

**Remark 5.3.** using the functional interpretation of the rationals, the logarithmic derivative of the zeta function looks like a sort of 2D discrete Dirichlet series:

$$\frac{d}{ds} \zeta(s) = \sum_{p \in P} \sum_{k \in N} k \ln(p)(p^k)^{-s} = \int_{\pi} \int_{\mathcal{N}} x^{-s} dk dm.$$

\(^2\)It is the thermodynamic energy in the primon model, where the zeta function is the partition function.

\(^3\)\(\mu \ast 1 = \delta\) in the convolution algebra of arithmetic functions.
On the other hand it is a weighted superposition of $1/(1 - \chi_p(s))$:

$$-\frac{d}{ds} \ln \zeta(s) = \sum_{p \in \mathbb{P}} \frac{\ln(p)}{p} \cdot \frac{1}{p^s - 1}.$$ (3)

Now let’s try to relate this “Lie-algebra/group” picture, which naturally fits into algebraic quantum groups theory (with its nice duality theory), with the “adelic picture”, but not “a la” Connes via moduli space of adeles mod ideles ($(K^\times, \cdot)$ acts on $(K, +)$, but the quotient “foliation” is bad, per Grothendieck’s advise :), but via an algebraic quantum duality, similar to the duality between universal enveloping algebra $U(g)$ and regular functions on the group $\mathcal{F}(G)$; perfect setup for the Quantum Groups machinery.

Now some additional pieces of the puzzle.

1) The p-adic tori $T_p = \mathbb{Q}_p/\mathbb{Z}_p$ fill the algebraic (“cyclotomic”) circle:

$$Q/Z \cong \oplus \mathbb{Q}_p/\mathbb{Z}_p.$$

2) The p-adic integers are quantum $\mathbb{Z}$-modules, i.e. deformations of the basic elementary abelian groups (see [12] for the classical case) (I avoid saying “finite fields”: they are Klein geometries, and the multiplication comes for free as the symmetries of the “discrete space of vectors” $(\mathbb{Z}/p\mathbb{Z}, +)$; just the set with a 1-D coordinate system on it):

$$Z_p = (F_p[[h]], \ast), \quad a = a_0 + a_1 p + ..., \quad b = b_0 + b_1 p + ...$$

$$a \ast b \leftrightarrow a_0 \ast b_0 = a_0 \cdot b_0 + c(a_0, b_0) h,$$

where the 2-cocycle $c$ is defined by the s.e.s.

$$\mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p^2\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}.$$ Alternatively, $\mathbb{Z}/p^2\mathbb{Z}$ can be first viewed as a 1st order deformation (central extension), but comparison with the above s.e.s.

Remark 5.4. A by-the-way question at this point: since the finite field $F_p$ is multiplicatively speaking the symmetries of the space $(\mathbb{Z}/p\mathbb{Z}, +)$:

$$(F_p^\times, \cdot) \cong \text{Aut}(\mathbb{Z}/p\mathbb{Z}, +),$$

is it the case that the filtration of the unique Galois extensions $F_{p^n}$ are the symmetries of the p-adic integers (as “quantum spaces”? (relation with Frobenius map ... later).

So, if p-adic integers are quantum modules, and we complete their quotient fields to the projective p-adic curve:

$$Q_p \twoheadrightarrow \mathbb{Z}_p \times \mathbb{Z}_p / \sim = P^1 \mathbb{Z}_p,$$

is this a “good” object to rely on to study duality of the rationals?

Is this projective space “spherical” (trivial cohomology), like the Riemann sphere, or because of the $(p - 1)$-roots of unity has a non-trivial “genus”?

Remark 5.5. Here we should think of Hessel’s analogy of p-adic numbers as power series, hence as “meromorphic functions”; relation to $SL_2(\mathbb{Z})$? what is the theory of p-adic modular functions?

Remark 5.6. The relations between rationals $Q \subset P^1 \mathbb{Z}$, integer Moebius transformations ($SL_2(\mathbb{Z})$), continued fractions and Farey fractions is definitely intriguing; what happens when we embed the rationals in the algebraic adeles?
6. The Statistics of Numbers

In the primon gas model [18, 17] the primes are fundamental states of a quantum (discrete) system with energy levels \( E_p = \ln(p) \), the rationals are populations with occupation numbers given by the fundamental theorem of arithmetic:

\[
n = \prod p^{N(p)}, \quad E_n = \sum N(p)E_p = \ln(n)
\]

and the resulting partition function is the Riemann zeta function:

\[
Z(\beta) = \sum e^{\beta E_n} = \zeta(s),
\]

where the inverse temperature is the analytic parameter \( \beta = s \).

Then the Riemann-Mangoldt exact formula expresses the thermodynamic energy (or spectrum \( E = h\nu \)) in two ways (see equations 2 and 3), as a trace (eigenvalues or diagonal elements):

\[
\sum \ln p \frac{1}{p^{s-1}} = < E > = \sum_{c \in \mathbb{C}_P} \frac{1}{s-c}.
\]

As noted before, the RHS is the sum of sources of the “electromagnetic flow” (conformal geometry in 2D), while the LHS is reminiscent of the Planck’s Law of radiation, with the “density” of lower energy states from the Prime Number Theorem

\[
\frac{1}{\pi(p)} \sim \frac{\ln p}{p}
\]

and spectral radiance [20] (\( \nu \) being the nth spectral line):

\[
B_{\nu}(T) \sim \frac{1}{e^{\beta E_{\nu}} - 1} = \frac{1}{p^{s-1}}.
\]

The relation becomes (with \( s = \beta \) for \( T \) and the number of nodes \( p \) in place of the frequency \( \nu \)):

\[
\sum_{p \in \text{States}} \frac{1}{N(p' < p)} B_p(s) = \sum_{c \in \text{Sources}} \frac{1}{s-c}.
\]

That such a connection with Planck’s Law of Radiation (QED interaction) exists is compatible with the many instances when the zeta values appear as values of Feynman integrals in QED, and in connection with the fine structure constant which “should” be a partition function itself [?], related to the prime zeta function:

\[
P(s) = \sum_{p \in \mathbb{P}} p^{-s}.
\]

The suggestion of Hilbert and Polya, for interpreting the RH problem in the context of a Hilbert space and selfadjoint operator, should be converted into an investigation how to formulate QED and the Hopf algebra approach to renormalization (Connes-Kreimer), as a problem of Number Theory, via rooted trees decorated with \( SL_2(\mathbb{Z}) \) operators, in a similar way to Kontsevich work on the Formality Theorem.

This is where the correspondence between primes (basic finite fields as discrete dynamical systems) and rooted trees should show its use. Ultimately, if “Fundamental Physics is Number Theory”, the H-atom QED interaction (graphs on genus 2 Riemann surfaces) as a fundamental quantum system should be the physics model extending the Riemann gas / Primon model.
7. “Conclusions”

So, reverse engineering the exact formula (Riemann-Mangoldt or Weil), should lead to the actual duality between the “original objects”.

Since $\zeta'/\zeta$ (or its “fermionic” counterpart $\zeta^- = DS(\mu)$), looks like a Weierstrass “cos” function (even part of $exp$), it should be a “periodic coordinate” on some lattice quotient, i.e. duality: s.e.s. or group/convolution algebra. It probably is an AQG duality, when taking into account the approaches of Meyer, Burnol etc., with central object the “orbit integral” / periodization operator $Z$, a sum of multipliers (dual Hopf algebra).

Moreover, the completed zeta function $\xi(s)$, i.e. the more “important” one, since it satisfies the functional equation, and no longer has the trivial zeros, is in fact the Mellin transform (MT) of this periodization operator, denote in [?] p5 by $\theta$:

$$\theta_f(y) = \sum_{n \in \mathbb{Z}} f(y \cdot n), \Gamma_f = MT(f) = \frac{1}{2} \Gamma(s/2)/\pi^{s/2},$$

$$MT(Av(\theta)) = \Gamma_f(s) \cdot \zeta(s) = \xi(s),$$

where $Av$ is the usual operator averaging a function at jump discontinuities (Fourier convergence / Gibbs phenomenon). This “complication” can be avoided if restricting to $\mathbb{N}$ (and $\mathbb{Q}^+$).

The relation $MT(Z(f)) = DS(1) \cdot MT(f)$ leads to an “expectation value” for the periodization operator (orbit integral):

$$DS(1) = \frac{MT(Z(f))}{MT(f)}.$$  

It is tempting to interprete the Dirichlet series as a periodization also:

$$DS(1) = \sum_{n \in \mathbb{N}} (0 + n)^{-s},$$

sort of a renormalization factor ... 

So a possible task is to “extract” the AQG content “a la” Van Daele, from the presentations using distributions, “a la” Weil.

Since the RM/W exact formula exhibits a conjugation symmetry (like taking an even part), which is natural since it comes from the “Weierstrass zeta-function” [10]:

$$\frac{1}{2} \sum_{\rho=1/2+i\gamma} FT(g)(\gamma) = Re(h(i/2) + <\hat{g}, ?? >) - ??^4$$

it is natural to look for the “other” part (“sin”, $\mathcal{P}'$?).

The “Hodge/Jacobi/Abel” like isomorphism relating the primes as “minimal cycles”, and zeroes as eigenvalues of the Laplace operator $\Delta = (d + d^*)^2$ is not yet known, but it should belong to a “global object”: gluing the p-adic quantum spaces $\mathbb{Z}_p$ (with $\mathbb{F}_p$ as tangent spaces) as a formal manifold a la Kontsevich (he defines the local version: formal pointed manifold, and was looking with Soibelman for a way to glue them).

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[^4]: What is the intrinsic meaning of these terms?

[^5]: The Mangold lambda function is in fact a measure on the set of primes, such that the Chebyshev’s function is an integral of a bounded region, when rationals are interpreted as functions $n : P \rightarrow \mathbb{Z}$. 

Since there is only one fundamental object in the physics world, I expect that a Feynman Path Integral algebraic model of the hydrogen atom (qubit from Quantum Computing picture of the Universe $SU_3$ vs. $SO_3$ etc.), will be the looked for quantum dynamical system yielding the zeta function as a partition function (like in the purely formal way in the Primon model), and clarify what the zeroes are (closely related to the usual quantum numbers of the hydrogen spectrum; e.g. principal quantum number $n$ etc.).

Now the prime numbers, or rather the tangent spaces $F_p$ of the Fractal Formal Manifold $Ab_f$, as a dynamical system with discrete flow $Aut_{Ab}$, has a quantum deformation (graded fractal hierarchy of structure) via $p$-adic numbers as local charts, overlapping via the “transition functions” defined (somehow) by their common local symmetries:

$$Aut(F_p, +) \leftarrow \oplus_{q^k|\gcd(\phi(p), \phi(p'))} \mathbb{Z}/q^k(q) \overleftarrow{\to} Aut(F_p')$$

with $q$ prime and $k(q)$ the maximal exponent.

The non-canonical duality of $\mathbb{Z}/p^k\mathbb{Z}$ could enter into the picture as a state-sum.

From duality, somehow, one should derive the primes-zeroes correspondence (a la Hodge).

At least exploring some of these ideas could be an interesting research journey :) 

References

[16] LI, The rational numbers as an algebraic quantum group, lecture notes, web.

6Ultimate Physics Theory is Number Theory, right? [11]