RESEARCH STATEMENT

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Abstract. My current research is related to the Riemann Hypothesis, because of its importance to physics: primes and Riemann zeros are in a particle-wave duality, far beyond Pythagoreans expectations; it is the music of adelic strings ...

The path adopted leads to adelic analysis, a new territory analog to complex analysis, but of algebraic essence. The journey is at the same time fruitful for elementary mathematics; it suggests projects that students of various levels (K-12, UG, G) can comprehend, enjoy and play with.

Contents

1. Introduction 1
2. Current Research 2
2.1. Research on the Riemann Hypothesis 2
2.2. New area: Adelic Analysis 3
2.3. Projects for students 4
3. Methodology 4
4. Conclusions 4
5. Appendix - Two examples 5
5.1. The Alchemy of Primes: a physics analogy in Number Theory 5
References 9

1. Introduction

It is hard separating research from teaching, methodology from pedagogy, and service to the community; integrating them comes naturally, as we mature

After a brief overview, research oriented, I would like to include two concrete examples, since “a theory is two good examples”.

My research in mathematics is related to quantum physics, following a top-down approach from ideas to the development of new areas in mathematics. It is influenced
by modern trends in mathematical-physics: deformation theory, applications of Lie-Klein ideas, and emphasis on p-adic numbers from the deformation theory viewpoint, towards Number Theory as the language of the “Ultimate Physics Theory”.

2. Current Research

From Physics to Number Theory and Back My research in deformation theory, Feynman Path Integrals and renormalization, led to the study of the Riemann zeta function [1]. Categorification and going back to physics (Physification), are working principles which provide guidance and meaning to mathematics investigations.

As a general trend, the mathematics of p-adic numbers is becoming more and more important. In mathematics, they allow unifying the characteristic $p$ mathematics with the usual “torsion-less” mathematics of characteristic zero, right into the world of graded mathematics (e.g. quantum groups etc.). This is a direction initiated with Hensels idea of developing Number Theory as part of p-adic analysis. In mathematics-physics it is also a motto: “Ultimate Physics Theory” is Number Theory (see e.g. Volovichs work). Specifically, I project that the future string theory will be formulated in a discrete form with p-cycles (Z-Spec) as “elementary strings” (vibrating structures), with the Riemann spectrum (R-Spec) as their energy spectrum, in an algebraic-geometric duality between primes and zeros.

2.1. Research on the Riemann Hypothesis. Working on the Riemann Hypothesis, following Grothendiecks advise, one needs to develop the theory first. Taking into account the new trends in mathematical-physics, deformation theory, quantum groups, TQFTs etc., the continuation of Tates and Iwasawas approach based on adelic harmonic analysis, should be away from Weils choice of distributions, and in the direction of implementing the Z-Spec / R-Spec duality as a duality in the sense of algebraic quantum groups [8, 9].

![Figure 1. Research directions regarding the Riemann Hypothesis](image-url)
Hensel’s analogy between \( p \)-adic numbers and analytic functions, and his insight that Number Theory (finite order deformations) should be developed as part of \( p \)-adic analysis [10] (via deformation theory [9]), can be extended towards an adelic analog of Complex Analysis.

### 2.2. New area: Adelic Analysis.

To be brief, the core of Complex Analysis is \( U(1) \)-equivariance and “Cauchy Path Integration” (analytic continuation). This is the common ground where Quantum Physics (Feynman Path Integral quantization) meets Riemann Surfaces, analytic continuation, Riemann zeta function (deformation theory).

Adelic analysis regards adeles as a collections of Laurent series (localizations) of “meromorphic functions” on a fictitious “analytic space” (for non-commutative geometry, you don’t need a space etc.). This contrasts with the usual “automatic” generalization of real analysis to \( p \)-adic analysis, and its use to \( p \)-adic differential equations, e.g. in the work of Dwark to prove the Weil conjectures.

The key idea is to consider finite fields extensions (algebraic-geometric completion) together with \( p \)-adic completion (topologic/ deformation theory), and study the adeles themselves as an analog of complex analysis:

![Figure 2. Adelic “Analysis” (Algebraic-Geometry)](image_url)

The “cast of characters” is: an adele is the collection of Laurent series at various “places”, \( Q^\times \) “are” Mobius transformations, \( \hat{Q} = \text{Hom}(Q, S^1) \) “are” Feynman amplitudes \( \exp(\int_\gamma \! f(z)dz) \) (Cauchy integral as action functional), cycles \( \gamma \) are boundaries of Zarisky open sets; zeta integrals occur as a Hilbert space formalism of multiplicative characters (transformations):

\[
K(A, B) = \int_{\gamma \in \text{Hom}(A,B)} e^{i \int_\gamma \! f(z)dz},
\]
where $\alpha = f(z)dz$ is the 1-form associated to a (fictitious) meromorphic function corresponding to an adele.

The main point is that Cauchy Integral Theorem holds when a class of linear functionals are compatible with the *homotopy relation of a groupoid*, here the tautological bundle $A^x \times A$ (think: $A^x \to Aut_{Ab}(A,+)$), the central piece of harmonic analysis in the approach of Bost-Connes [11, 13] for the spectral interpretation of the Riemann zeros.

### 2.3. Projects for students

This work is prone to some fun projects accessible to students, projects emerged from computational explorations of the Riemann frequencies (spectrum) with SAGE (see also Annex: Example 2), while using statistical methods to detect their underlying structure, e.g. statistical correlations/ resonances, dual to the correlations of primes due to common symmetries (Appendix: Example 1.5.1 - see also NSF GP 2014 [5]).

Other equally fun and simple projects involve the study of the “alchemy” of prime numbers (Appendix: Example 2.5.2), involving their internal structure (see Pratt trees, 1975 and L.I.s POSet of Primes 2011 [3]), ignored for so long, while pursuing traditional paths.

### 3. Methodology

Top-down design is how a “bird” mathematician [2] closes on a new result, starting from a good idea and insight on the problem; it contrasts with the bottom-up constructive methods of analysis for example.

We venture a few ideas regarding what mathematics currently needs: 1) Several layers/levels of interfaces separated from implementations (details); 2) A deeper hierarchy of languages, from math-code to “story telling”; 3) A graphical and iconic interface for ideas and maps of the territory (birds-eye-view).

With a good working knowledge of the history of mathematics, taught many times, always with pleasure, I develop mathematics and advise students in the direction of the new trends, away from conservative areas and tools, especially when historically proven unsatisfactory.

### 4. Conclusions

I am a life-long learner, who needs to expand in a new research oriented mathematics department, ready to continue learning, climbing course-by-course, and enjoying debates about mathematics in general and the mathematics profession in particular.

My research agenda in mathematical-physics takes into account the present needs of sciences, and mathematics in particular, in its historical development. It is influenced by how knowledge in society evolves, and uses a personalized research methodology
(top-down design, interface-implementation structure), while being open to new challenges. Although used to work independently, I enjoy collaborations (like the years when I was a software project manager), and I am eager to work in a team if available.

5. Appendix - Two examples

5.1. The Alchemy of Primes: a physics analogy in Number Theory. Number Theory, “The Queen of Mathematics”, is in some sense the “chemistry of primes”; they are the building blocks, e.g. $6=2\times3$ and $Z_6=Z_2\times Z_3$ etc. (from elementary NT and abstract algebra level, to Fourier Analysis on finite abelian groups, graphs and zeta functions and more; see for example Audrey Terras work on the subject, or Christian Mercat Discrete Riemann Surfaces, for a few possible ramifications of these ideas).

The “alchemy” of primes refers to the internal structure of generators (symmetries) of basic finite fields, which leaves as a shadow the partial order on the set of primes, so important in connection with primality tests, as observed earlier by Pratt (1975).

5.1.1. Euclid’s “trick”. To exemplify the use of analogies and a graphical interface as pedagogical tools when teaching mathematics, we build on the analogy with nuclear physics, and present Euclid’s “trick” $N = p_1 \ldots p_{k+1}$, used to prove the infinity of prime numbers), as a fusion-fission process: n-primes in (Input) “collide” (product) and “interact” (“+1”), resulting in a new product which may be “unstable”, factoring into other primes $q_1, q_l$ (Output):

![Figure 3. Primes “nuclear reaction”: $p_1 \cdot p_2 \ldots p_k + 1 = q_1 \cdot \ldots \cdot q_l$](image)

If taking all small primes less then a cut-off (primorial $p_n\#$) then “new” primes emerge. The first “non-stable” such interaction is

$$p_6\# + 1 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 59 \cdot 509,$$

as students readily reported using the web (a student had a laptop), and team work (History of Mathematics class).
5.1.2. *The inverse problem.* The “inverse problem” consists in determining the primes which yield a given output \( q \); but this leads at factorizing \( q - 1 \), i.e. the first level in its Pratt tree. Iteration of this process leads to the full Pratt tree and the introduction of the POSet or primes. For example

\[
 t(181) = B^+(\bullet, t(2), t(2), t(3), t(3), t(5)), \quad t(3) = B^+(\bullet), \quad t(5) = B^+(\bullet, \bullet).
\]

where \( B^+ \) is the operator which adjoins a common root to the input rooted trees. Of course the labels \( 2, 3 \) and \( 5 \) are not necessary; the corresponding “prime values” of the labels can be reconstructed recursively, from terminals going up.

This is a direct invitation for the students to investigate, in search of new facts and results, which are very much accessible [3].

5.2. *The Music of Primes: Integrating Mathematics, Statistics and Computer Science.* The duality between primes and Riemann zeta zeros is a rich source of topics for all levels, and all mathematics flavors; also at the foundations of physics, as it becomes increasingly evident.

The duality can be expressed via the Riemann-Mangoldt-Weil exact formula, or using distributions as sketched below [5]. When compared with the Poisson Summation Formula for real distributions [7], p.1, with \( \hat{\delta}(n) = \exp(2\pi int) \):

\[
\sum_{n \in \mathbb{Z}} \delta(n) = \sum_{n \in \mathbb{Z}} \hat{\delta}(n),
\]

one would like to have a similar interpretation relating the Dirac “harp” of primes \( D_P \) with that of the Riemann spectrum \( D_\Theta \), Equation 1 \(^1\), coming from an actual Fourier isomorphism, probably of algebraic quantum groups associated to the adelic duality.

5.2.1. *From primes to zeros.* The following diagram represents an approximation of the Fourier transform \( \hat{D}_F^C \) = \( \sum_{p,n,p^\alpha < C} \frac{1}{\sqrt{p}} d_{log_p \alpha}(t) \), using a cut-off \( C = 3 \) (similarly \( P_C(\theta) \) is the partial sum with terms \( p^\alpha \leq C \)). The peaks of the Fourier Transform

\(^1\)A Dirac comb implies equal amplitudes.
From [6], part IV, p.107

seem to converge to the zeta zeros $\theta_n$ (see Fig.2). In the distributional sense:

$$\lim_{C \to \infty} \mathcal{D}_C^P(\theta) = \lim_{C \to \infty} \sum_{p^n < C} \frac{\log p}{\sqrt{p^n}} \cos (n \log p \theta) = \sum_n d_{\theta n}(\theta) = D_\Theta(\theta).$$

5.2.2. From zeros to primes. In the other direction, from the Riemann spectrum to the primes, the Inverse Fourier Transform yields in a similar way, via approximations (see Fig.4):

$$\mathcal{D}_\Theta(\log s) = \sum_n \cos (\theta_n \log s) \overset{w}{=} \sum_{p \in P, n \in N} \frac{\log p}{\sqrt{p^n}} d_{\rho n}(s) = \int \int_{P \times N} \frac{1}{\sqrt{p^n}} d_{\rho n}(s).$$

The maxima of the $C = 1000$ approximation $T_C(\theta)$ are localized at power of primes, the limit being $\mathcal{D}_P \circ e^t$. With $s = e^t$ we obtain

$$\mathcal{D}_\Theta = \mathcal{D}_P,$$

as expected.

5.2.3. Integrating Mathematics, Statistics and Computer Science. The main point is that a study of Riemann zeros can be done using available software, e.g. SAGE, stimulating programming skills for mathematical applications. The study can then branch towards a statistics study of the Riemann zeros, in search for the underlying structure guaranteed by their low entropy.

For example their distribution on the unit interval can be easily computed reproducing the work by Ford a.a. [4]. It corresponds to $\alpha = \log p/(2\pi) \cdot 1/3$. Taking $q = 3$
The correlations between the “random variables” \( X_p = \{ p^{i\gamma} | \gamma \in R - Spec \} \) (mapping the Riemann spectrum on the unit circle), for various primes, can be also investigated. It aims to detect a “mirror symmetry” of the correlation of primes due to common symmetries suggested by the POSet structure, via the “hidden” duality:

\[ \alpha = \frac{1}{3} \cdot \frac{\log(2)}{2\pi} \] - generated by the PI using SAGE
Figure 7. Correlation coefficient $\text{co}(X_{31}, X_p)$ for the first 100 primes $p \neq 31$.

The “resonances” (correlation coefficient $> 0.15$), occur at the primes:

$p_{36} = 157, \, 2^2 \cdot 3 \cdot 13, \quad p_{67} = 337, \, 2^4 \cdot 3 \cdot 7, \quad p_{77} = 397, \, 2^2 \cdot 3^2 \cdot 11$

$p_{78} = 401, \, 2^4 \cdot 5^2, \quad p_{79} = 409, \, 2^3 \cdot 3 \cdot 17, \quad p_{97} = 521, \, 2^3 \cdot 5 \cdot 13, \quad p_{98} = 523, \, 2 \cdot 3^2 \cdot 29$.

The factorization of $p - 1$ is provided to the right of each prime $p$, showing the structure of each $\text{Aut}_{Ab}(F_p, +)$.

Although no definite conclusion is yet available, the SAGE code is easy to write and gives the student the confidence of having a “grip” on the Riemann zeroes (at least for a friendly interrogation :).

5.2.4. Conclusions. In conclusion, we believe that a broad view of mathematics and its connections with other sciences is a beneficial, complementary alternative to the usual high level of specialization, and both can yield a high level of academic performance.

Regarding the Riemann Hypothesis, helping the student attack, or at least understand such a famous problem early in one’s career, can have beneficial consequences.

References


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